The amplification of a weak applied magnetic field by turbulence in fluids of moderate conductivity

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(Received 31 May 1961)

The effect of turbulence on an applied magnetic field is considered in the case when the magnetic Reynolds number R_m is large compared with unity but small compared with the ordinary Reynolds number R of the turbulence. When the applied field is sufficiently weak, it is argued that its effect on the velocity field is negligible. The equation for the field is then linear and its spectrum may be obtained throughout the equlibrium range of wave-numbers. It appears that the spectrum increases as $k^{\frac{1}{3}}$ up to a wave-number k_c marking the threshold of conduction effects, and falls off as $k^{-\frac{11}{3}}$ beyond k_c. The net effect of the turbulence is expressed in terms of an eddy conductivity equal to $R_m^{-\frac{1}{2}}$ times the electrical conductivity of the fluid. The effect of magnetic forces when these are not negligible is also tentatively considered.

1. Introduction

The behaviour of a magnetic field in a turbulent conducting fluid is largely determined by the relative magnitudes of the Reynolds number R and the magnetic Reynolds number R_m of the turbulence. These may be defined in terms of the root-mean-square velocity u' and a length L characteristic both of the energy-containing eddies and, it may be supposed, of any large-scale magnetic field disturbance (or magnetic eddies) that may be present. Thus

$$R = u'L/\nu, \tag{1.1}$$

and

$$R_m = 4\pi\mu\sigma u'L = u'L/\lambda, \tag{1.2}$$

where ν is the kinematic viscosity of the fluid, and μ, σ and λ are its permeability, conductivity and magnetic diffusivity, respectively. Throughout this work, we shall suppose that R is at least five or six orders of magnitude greater than unity.

When $R_m \gg R$ (i.e. $\lambda \ll \nu$), any weak random magnetic field is intensified by the action of the turbulence; indeed its mean-square value increases exponentially until magnetic stresses react back upon the velocity field (Batchelor 1950). At the other extreme, when $R_m < 1$, i.e. in a weakly conducting fluid, conduction effects are dominant at all length scales, so that any random field will rapidly decay to zero. Steady conditions are possible, however, if a large-scale magnetic field is maintained by externally applied electromotive forces. The turbulence will then give rise to small fluctuations in this field whose spectral properties will be closely related to the turbulent spectrum and whose level will be controlled by the small conductivity of the medium. Golitsyn (1960) has recently analysed the particular case of turbulence of a weakly conducting fluid in a uniform magnetic field, and has obtained the anisotropic spectrum of the small-scale field fluctuations that are induced.

There remains the possibility, which we investigate in this paper, that

$$1 \ll R_m \ll R. \tag{1.3}$$

This is the case of moderate conductivity which may well arise in problems of astrophysical and geophysical interest. The condition implies that $\lambda \gg \nu$, so that, according to Batchelor, random magnetic field perturbations decay to zero in the absence of electromotive forces. The reason for this is that conduction effects become important at a length scale l_c large compared with the length scale l_v of the smallest turbulent eddies at which viscous dissipation begins to predominate. The lines of force of any magnetic field disturbance on a length scale larger than l_c are, to a good approximation, carried with the fluid so that initial intensification may result. But such a field is presumably distorted by the smallest velocity eddies and broken up into components much smaller than l_c which must ultimately decay to zero through the predominant conduction effect. Again it appears that a steady spectrum can be maintained only if externally applied electromotive forces are present. We shall suppose that these generate, on the scale L, a magnetic field $\mathbf{H}_0(\mathbf{r})$ which is distorted by the turbulence, intensification through the stretching mechanism occurring at length scales larger than l_c, with conductive decay at length scales smaller than l_c.

It will be possible to neglect the back-reaction of the magnetic field on the velocity field provided the mean magnetic energy generated is small compared with the kinetic energy of those eddies whose scale is small compared with L, itself a factor $R^{\frac{1}{2}}$ less than the total kinetic energy of the turbulence (see §4). Since the equation for the magnetic field is linear, the magnetic energy will be proportional to $\overline{H_0^2}$ (and probably larger than $\overline{H_0^2}$ in view of the initial intensification). We shall therefore suppose that $\overline{H_0^2}$ is sufficiently small for us to neglect the back-reaction, although we may later examine, at least qualitatively, departures from this condition.

Under the conditions outlined above, the statistical properties of the small-scale motion, characterized by wave-numbers k large compared with $1/L$, are according to Kolmogorov's theory, steady, isotropic, and determined solely by the parameters ν and ϵ, the rate of dissipation of kinetic energy per unit mass. The justification for these claims is that R is large compared with unity. Since R_m is likewise large compared with unity, the statistical properties of the small-scale magnetic field are, to the same approximation, steady and isotropic, though they may depend upon certain field parameters (e.g. λ) in addition to ν and ϵ.

A problem closely related to that under consideration was studied by Batchelor, Howells & Townsend (1959) who obtained the spectrum, at high wave-numbers, of a dynamically passive scalar solute θ under the combined action of convection and diffusion at small Prandtl number. It is interesting to note that the magnetic-field spectrum that we shall obtain in the following sections is identical with the spectrum of $\nabla\theta$, although the underlying kinematical reasoning is not the same in the two cases, since $\mathbf{H}$ and $\nabla\theta$ do not satisfy the same equations.

It is perhaps worth stressing at the outset that we are assuming that the level of the magnetic spectrum can be controlled by ohmic conduction, even though this predominates only at higher wave-numbers, simply because the intensification through stretching of lines of force is associated with a decrease of scale, i.e. the magnetic energy may be increased but is necessarily directed towards the ohmic sink at high wave-numbers at the same time. Some other authors (e.g. Biermann & Schlüter 1951) have taken the view that the increase of field intensity at low wave-numbers may continue until equipartition with the kinetic energy at the same length scale is established. However, the results that we shall derive are entirely consistent with our assumption which we therefore retain with some confidence.

It is easy to see why ohmic conduction does not necessarily control the magnetic spectrum level in the case of high conductivity when $\lambda \ll \nu$. For in this case, the conduction length scale l_c is much smaller than the viscous length scale l_v, so that the magnetic field cannot be broken up by turbulent eddies into small components at which conduction dominates. Indeed it can only be broken down into loops of size $O(l_v)$ at which the small-scale straining motion is very efficient at intensifying the field. But to pursue this argument is not the purpose of the present paper.

2. The magnetic energy spectrum in the range $1/L \ll k \ll (\epsilon/\lambda^3)^{\frac{1}{4}}$

The viscous cut-off wave-number k_v, depending only on ϵ and ν, is well known to be, in order of magnitude,

$$k_v = l_v^{-1} = (\epsilon/\nu^3)^{\frac{1}{4}}. \tag{2.1}$$

Similarly the conduction cut-off k_c, which is small compared with k_v and cannot therefore depend on ν, can depend only on ϵ and λ and is therefore given by

$$k_c = l_c^{-1} = (\epsilon/\lambda^3)^{\frac{1}{4}}. \tag{2.2}$$

We shall suppose that

$$L^{-\frac{3}{4}} \ll k_c^{\frac{3}{4}} \ll k_v^{\frac{1}{4}}, \tag{2.3}$$

so that we may consider separately two subranges of the inertial range of wave-numbers,

$$\text{subrange } A: 1/L \ll k \ll k_c \quad \text{and} \quad \text{subrange } B: k_c \ll k \ll k_v, \tag{2.4}$$

recognizing that if either of the inequalities of (2.3) fails to obtain, the corresponding subrange simply shrinks to zero. Since ϵ is given by the semi-empirical relation

$$\epsilon = u'^3/L, \tag{2.5}$$

the condition (2.3) is tantamount to the condition

$$1 \ll R_m^{\frac{3}{4}} \ll R^{\frac{3}{4}}, \tag{2.6}$$

a more exacting requirement than we anticipated in (1.3). In subrange A the magnetic energy spectrum is in a state of dynamic balance under the influence of the convection and consequent stretching of lines of force alone; we shall proceed to determine its form in this section leaving until §3 the study of subrange B in which conduction plays an important part.

Let us start from the observation, first made by Batchelor, that in a fluid for which $\lambda = \nu$, the equations for the rates of change of magnetic field $\mathbf{H}$ and vorticity $\boldsymbol{\omega} = \nabla \wedge \mathbf{u}$ are formally identical, namely

$$\partial \mathbf{H}/\partial t + \mathbf{u}.\nabla\mathbf{H} = \mathbf{H}.\nabla\mathbf{u} + \lambda\nabla^2\mathbf{H}, \tag{2.7}$$

and

$$\partial \boldsymbol{\omega}/\partial t + \mathbf{u}.\nabla\boldsymbol{\omega} = \boldsymbol{\omega}.\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}. \tag{2.8}$$

Thus, if $\mathbf{H} = C\boldsymbol{\omega}$ at any time, where C is a suitably small dimensional constant, so that the neglect of the Lorentz-force term in (2.8) is justified, then $\mathbf{H}$ will remain equal to $C\boldsymbol{\omega}$ at all subsequent times. In this case therefore all statistical properties of $\mathbf{H}$ and $\boldsymbol{\omega}$ will be identical, and in particular their spectra will have the same dependence on wave-numbers for $k \gg L^{-1}$. If the turbulence arises from some instability of a large-scale flow whose vorticity coincides with an applied magnetic field, then it is clear that the small-scale vorticity and magnetic field that are generated must also coincide.

Now if it is true that the statistical properties of the small-scale motion and field are independent of the large-scale specification, then under statistically steady conditions, for $k \gg L^{-1}$, and in a fluid for which $\lambda = \nu$, the statistical properties of any magnetic-field distribution are presumably the same as those of the particular field that coincides with the vorticity field. Hence the spectrum $\Gamma_H(k)$ of $\mathbf{H}$, defined for an isotropic, solenoidal field by the equations

$$\overline{H_i(\mathbf{x})\,H_j(\mathbf{x}+\mathbf{r})} = \int \Gamma_{ij}(k)\,e^{i\mathbf{k}.\mathbf{r}}\,d\mathbf{k}, \quad \Gamma_{ij}(\mathbf{k}) = \frac{\Gamma_H(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{2.9}$$

has the dependence, like the vorticity spectrum,

$$\Gamma_H(k) \propto k^{\frac{1}{3}} \quad \text{for} \quad L^{-1} \ll k \ll k_c. \tag{2.10}$$

If now the ratio λ/ν is increased, so that k_c decreases relative to k_v, the magnetic-field spectrum will be modified, but only over that part of wave-number space in which conduction effects are relevant, i.e. $k > k_c$. The relation (2.10) is unlikely to be altered throughout its stated range, although we might anticipate a fairly rapid cut-off for $k > k_c$, due to the rapid smoothing out of small-scale variations by conduction.

Since the foregoing argument may not carry complete conviction, it may be as well to supplement it with another which leans less heavily on the analogue with vorticity. The following argument employs the vector potential $\mathbf{A}(\mathbf{r}, t)$ defined by

$$\nabla \wedge \mathbf{A} = \mathbf{H}, \quad \nabla.\mathbf{A} = 0. \tag{2.11}$$

It is readily shown, from Maxwell's equations and Ohm's Law that $\mathbf{A}$ satisfies the equation

$$\frac{\partial A_i}{\partial t} + u_j \frac{\partial A_i}{\partial x_j} = u_j \frac{\partial A_j}{\partial x_i} + \frac{\partial \phi}{\partial x_i} + \lambda\nabla^2 A_i, \tag{2.12}$$

where ϕ is the electrostatic potential. The curl of this equation gives equation (2.7) for $\mathbf{H}$. If we now multiply (2.12) scalarly by A_i, average over ensembles, and use the property of homogeneity (by which the divergence of any quantity vanishes on averaging), we obtain without difficulty

$$d\overline{A^2}/dt = -2\overline{A_i A_j\,(\partial u_i/\partial x_j)} - 2\lambda\overline{(\nabla A_i)^2}. \tag{2.13}$$

It is easily shown that the spectrum of **A**, $\Gamma_A(k)$, defined by a pair of equations similar to (2.9), is by virtue of (2.11) related to $\Gamma_H(k)$ by

$$\Gamma_H(k) = k^2\Gamma_A(k). \tag{2.14}$$

It is reasonable to suppose that $\Gamma_H(k)$ varies as some power n of k in subrange A. We shall prove that if n lies between -1 and 1, then it necessarily has the value $\frac{1}{3}$. For in these circumstances, $\frac{1}{2}\overline{H^2} = \int_0^\infty \Gamma_H(k)\,dk$ is determined largely by values of $\Gamma_H(k)$ in the neighbourhood of $k = k_c$, whereas $\frac{1}{2}\overline{A^2} = \int_0^\infty \Gamma_A(k)\,dk$ is determined by values of $\Gamma_A(k)$ in the neighbourhood of $k = L^{-1}$ since $\Gamma_A(k)$ by hypothesis falls off more rapidly than k^{-1} for $k \gg L^{-1}$. In this sense, it is true to say that the wave-number ranges determining $\overline{A^2}$ and $\overline{H^2}$ do not overlap, provided R_m is large enough. Let us denote by χ the total rate of generation of contributions to $\overline{A^2}$ (or '$\overline{A^2}$-stuff') including generation by electromotive forces (not represented in equation (2.13)) and by interaction with the turbulence, represented by the term

$$-G\{\mathbf{A}\} = -2\overline{A_i A_j(\partial u_i/\partial x_j)} \tag{2.15}$$

of equation (2.13). Under steady conditions, equation (2.13) then gives

$$\chi = 2\lambda\overline{(\nabla A_i)^2}. \tag{2.16}$$

Thus, $\overline{A^2}$-stuff is generated at a rate χ at wave-numbers of order L^{-1}, and is destroyed at a rate χ at wave-numbers larger than k_c. It is therefore transferred at a rate χ through the spectrum which is therefore determined in subrange A solely by the parameters χ and ϵ. The dependence of $\Gamma_A(k)$ on χ must be mere proportionality because of the linearity of the equation for **A**, and dimensional analysis now gives $\Gamma_A(k) \approx (\chi/\epsilon^{\frac{1}{3}})\,k^{-\frac{5}{3}}$, so that $\Gamma_H(k) \approx (\chi/\epsilon^{\frac{1}{3}})\,k^{\frac{1}{3}}$.

Now n cannot be less than -1; for if so, we could apply the above argument to both $\Gamma_H(k)$ and $\Gamma_A(k)$, proving that both these spectra have the dependence $k^{-\frac{5}{3}}$, contrary to equation (2.14). Moreover, it is extremely unlikely that n should be greater than 1; no physical argument can be found to support such a rapid increase with k. Hence we are again led to the result (2.10).

It is interesting to note the resemblance between equation (2.13) and the equation for rate of change of $\overline{H^2}$ derived from equation (2.7), namely

$$d\overline{H^2}/dt = G\{\mathbf{H}\} - 2\lambda\overline{(\nabla H_i)^2}, \tag{2.17}$$

where $$G\{\mathbf{H}\} = 2\overline{H_i H_j(\partial u_i/\partial x_j)}.$$

The term $G\{\mathbf{H}\}$, representing the generation of magnetic energy by the turbulence, is of vital importance in determining the spectral properties of **H**. This is because considerable vorticity is associated with the wave-number region near k_c (since $k_c \gg L^{-1}$) in which the magnetic energy is concentrated. The same cannot be said of the term $-G\{\mathbf{A}\}$ in relation to the **A**-spectrum, since there is very little vorticity associated with wave numbers of order L^{-1} at which $\Gamma_A(k)$ is maximal, a comment best expressed by the inequality

$$G\{\mathbf{A}\}/G\{\mathbf{H}\} \ll \overline{\mathbf{A}^2}/\overline{\mathbf{H}^2}. \tag{2.18}$$

3. The magnetic energy spectrum in the range $k_c \ll k \ll k_v$

In this section we shall follow the method of Batchelor *et al.* (1959) to determine the spectrum in subrange B. The Fourier transforms of the fields $\mathbf{u}$ and $\mathbf{H}$ may be defined by the equations

$$\mathbf{u}(\mathbf{r}) = \int \mathbf{p}(\mathbf{k})\, e^{-i\mathbf{k}.\mathbf{r}}\, d\mathbf{k}, \quad \mathbf{H}(\mathbf{r}) = \int \mathbf{q}(\mathbf{k})\, e^{i\mathbf{k}.\mathbf{r}}\, d\mathbf{k}, \tag{3.1}$$

where $\mathbf{p}(\mathbf{k})$ and $\mathbf{q}(\mathbf{k})$ have the solenoidal property

$$\mathbf{k}.\mathbf{p}(\mathbf{k}) = \mathbf{k}.\mathbf{q}(\mathbf{k}) = 0. \tag{3.2}$$

The Fourier coefficients $p_i(\mathbf{k})$ and $q_i(\mathbf{k})$ are related to the kinetic-energy spectrum $E(k)$ and the magnetic-energy spectrum $\Gamma(k)$ (dropping the suffix H) by the equations

$$\overline{p_i(\mathbf{k})\, p_j^*(\mathbf{k})} = \frac{E(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \quad \overline{q_i(\mathbf{k})\, q_j^*(\mathbf{k})} = \frac{\Gamma(k)}{4\pi k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right), \tag{3.3}$$

where the star indicates a complex conjugate.

In terms of the Fourier coefficients, equation (2.7) may be written

$$\frac{\partial q_j(\mathbf{k})}{\partial t} + i\int k'_i p_i(\mathbf{k}-\mathbf{k}')\, q_j(\mathbf{k}')\, d\mathbf{k}' = i\int k''_i q_i(\mathbf{k}-\mathbf{k}'')\, p_j(\mathbf{k}'')\, d\mathbf{k}'' - \lambda k^2 q_j(\mathbf{k}). \tag{3.4}$$

In the integral on the right-hand side, it is expedient to change the variable of integration by writing $\mathbf{k}' = \mathbf{k} - \mathbf{k}''$, $d\mathbf{k}' = -d\mathbf{k}''$. Using the fact that $k'_i q_i(\mathbf{k}') = 0$, equation (3.4) becomes

$$\frac{\partial q_j(\mathbf{k})}{\partial t} = -i\int [k'_i p_i(\mathbf{k}-\mathbf{k}')\, q_j(\mathbf{k}') + k_i p_j(\mathbf{k}-\mathbf{k}')\, q_i(\mathbf{k}')]\, d\mathbf{k}' - \lambda k^2 q_j(\mathbf{k}). \tag{3.5}$$

The immediate aim is to convert this equation to one relating $\Gamma(k)$ and $E(k)$ by means of equations (3.3). The result of this manipulation will be found in equation (3.10).

Let us focus attention on values of $\mathbf{k}$ in subrange B, i.e. those satisfying

$$k_c \ll k \ll k_v. \tag{3.6}$$

The integral in (3.5) is over all wave-number space; but since we expect that $q_j(\mathbf{k}')$ will decrease rapidly as k' increases beyond k_c, because of the predominating influence of conduction, the integral will be dominated by the contribution from the range $k' < k_c$, that is, using (3.6), from values of $\mathbf{k}'$ satisfying $k' \ll k$, or equivalently

$$k' \ll |\mathbf{k} - \mathbf{k}'|. \tag{3.7}$$

This argument was given by Batchelor *et al.*, only the first summand of the integral in (3.5) appearing in their context. They moreover gave arguments which also carry over to the present case to show that the time derivative in (3.5) just balances the small contribution to the integral from values of $\mathbf{k}'$ near $\mathbf{k}$, and that both may therefore be neglected.

It then follows from (3.5) that

$$\begin{aligned}\lambda^2 k^4 \overline{q_j(\mathbf{k})\, q_j^*(\mathbf{k})} = \iint \{ & k_i' k_k''\, \overline{p_i(\mathbf{k}-\mathbf{k}')\, p_k^*(\mathbf{k}-\mathbf{k}'')\, q_j(\mathbf{k}')\, q_j^*(\mathbf{k}'')} \\ & + k_i' k_k\, \overline{p_i(\mathbf{k}-\mathbf{k}')\, p_j^*(\mathbf{k}-\mathbf{k}'')\, q_j(\mathbf{k}')\, q_k^*(\mathbf{k}'')} \\ & + k_i k_k''\, \overline{p_j(\mathbf{k}-\mathbf{k}')\, p_k^*(\mathbf{k}-\mathbf{k}'')\, q_i(\mathbf{k}')\, q_j^*(\mathbf{k}'')} \\ & + k_i k_k\, \overline{p_j(\mathbf{k}-\mathbf{k}')\, p_j^*(\mathbf{k}-\mathbf{k}'')\, q_i(\mathbf{k}')\, q_k^*(\mathbf{k}'')}\}\, d\mathbf{k}'\, d\mathbf{k}''. \end{aligned} \quad (3.8)$$

This double integral is dominated by contributions from the range for which $(k', k'') \ll (|\mathbf{k}-\mathbf{k}'|, |\mathbf{k}-\mathbf{k}''|)$, and in this range the statistical connexion between the p's and q's of (3.8) is slight. Hence, for example,

$$\overline{p_i(\mathbf{k}-\mathbf{k}')\, p_k^*(\mathbf{k}-\mathbf{k}'')\, q_j(\mathbf{k}')\, q_j^*(\mathbf{k}'')} \doteqdot \overline{p_i(\mathbf{k}-\mathbf{k}')\, p_k^*(\mathbf{k}-\mathbf{k}'')}\;\; \overline{q_j(\mathbf{k}')\, q_j(\mathbf{k}'')}.$$

The orthogonality of the coefficients,

$$\overline{p_i(\mathbf{k}-\mathbf{k}')\, p_k^*(\mathbf{k}-\mathbf{k}'')} = \overline{p_i(\mathbf{k}-\mathbf{k}')\, p_k^*(\mathbf{k}-\mathbf{k}')}\, \delta(\mathbf{k}'-\mathbf{k}''),$$

now allows a trivial integration throughout the $\mathbf{k}''$-space. Further, since $k' \ll k$, and $p_i(\mathbf{k})$ decreases slowly compared with $q_i(\mathbf{k})$ in the range (3.6), we may replace $p_i(\mathbf{k}-\mathbf{k}')\, p^*(\mathbf{k}-\mathbf{k}')$ by $p_i(\mathbf{k})\, p_k^*(\mathbf{k})$, which may now be brought outside the integral. These simplifications reduce (3.8) to the form

$$\begin{aligned}\lambda^2 k^4 \overline{q_j(\mathbf{k})\, q_j^*(\mathbf{k})} = \overline{p_i(\mathbf{k})\, p_k^*(\mathbf{k})} & \int k_i' k_k'\, \overline{q_j(\mathbf{k}')\, q_j^*(\mathbf{k}')}\, d\mathbf{k}' \\ & + 2k_i \overline{p_j(\mathbf{k})\, p_k^*(\mathbf{k})} \int k_k'\, \overline{q_j(\mathbf{k}')\, q_i^*(\mathbf{k}')}\, d\mathbf{k}' \\ & + k_i k_k \overline{p_j(\mathbf{k})\, p_j^*(\mathbf{k})} \int \overline{q_i(\mathbf{k}')\, q^*(\mathbf{k}')}\, d\mathbf{k}'. \end{aligned} \quad (3.9)$$

Now

$$\int k_k' \overline{q_j(\mathbf{k}')\, q_i^*(\mathbf{k}')}\, dk' = \overline{H_i \frac{\partial H_j}{\partial x_k}} = \overline{H_j \frac{\partial H_i}{\partial x_k}} = \frac{1}{2} \overline{\frac{\partial}{\partial x_k} H_i H_j} = 0, \text{ by homogeneity.}$$

Hence the second term of (3.9) vanishes. Also

$$\int k_i' k_k' \overline{q_j(\mathbf{k}')\, q_j^*(\mathbf{k}')}\, d\mathbf{k}' = \overline{\frac{\partial H_j}{\partial x_i} \frac{\partial H_j}{\partial x_k}} = \tfrac{1}{3} \overline{(\nabla H_j)^2}\, \delta_{ik},$$

by the isotropy of the small-scale magnetic field, and

$$\int q_i(\mathbf{k}')\, q_k^*(\mathbf{k}')\, d\mathbf{k}' = \overline{H_i H_k} = \tfrac{1}{3} \overline{H_j^2}\, \delta_{ik}.$$

Using these results and equations (3.3), the relation between $E(k)$ and $\Gamma(k)$ follows in the form

$$\lambda^2 k^4 \Gamma(k) = \tfrac{1}{3} E(k)\, \overline{(\nabla H_j)^2} + \tfrac{1}{3} k^2 E(k)\, \overline{H_j^2}. \quad (3.10)$$

The interpretation of this equation is as follows. Part of the magnetic energy in the wave-number range $(k, k+dk)$ is derived from the simple interaction of velocity components of wave-number k with an effectively uniform field gradient $\{\overline{(\nabla H_j)^2}\}^{\frac{1}{2}}$. This is exactly as for the scalar spectrum, and corresponds to a transfer of magnetic energy down the spectrum. The other contribution to $\Gamma(k)$ is a direct transfer from kinetic energy of wave-number k through the action of velocity gradients (i.e. rates of strain) on an effectively uniform field of magnitude $\{(\overline{H_j^2})\}^{\frac{1}{2}}$.

Now we may estimate the value of $(\overline{\nabla H_j})^2$ in terms of $\overline{H_j^2}$ as follows.

$$\tfrac{1}{2}(\overline{\nabla H_j})^2 = \int_0^\infty k^2\Gamma(k)\,dk = \int_0^{k_c} k^2\Gamma(k)\,dk + \int_{k_c}^\infty k^2\Gamma(k)\,dk.$$

In the first integral the integrand is increasing throughout most, if not all, of the range; in the second it is decreasing throughout the range. The integrals converge at the origin and at infinity, respectively. They are therefore determined by the

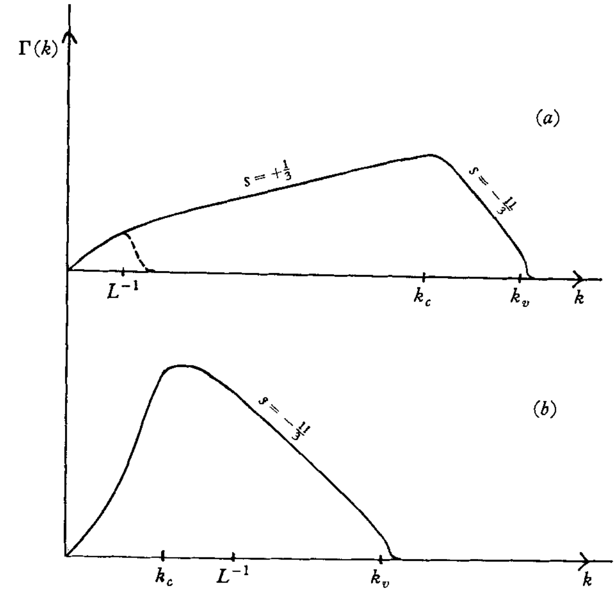

FIGURE 1. Wave-number dependence of magnetic energy spectra (on logarithmic scale), (a) when $1 \ll R_m \ll R$, and (b) when $R_m < 1$. The slope is represented by the letter s. The dotted curve in (a) represents the cut-off of the spectrum of the applied magnetic field $\mathbf{H}_0(\mathbf{r})$.

value of the integrand at k_c and may both be approximated by the expression $k_c^3\Gamma(k_c)$, neglecting numerical constants. Similarly $\frac{1}{2}\overline{H_j^2} = \int \Gamma(k)\,dk$ may be approximated by $k_c\Gamma(k_c)$. Hence $(\overline{\nabla H_j})^2 \approx k_c^2\overline{H_j^2}$, so that the second term on the right of (3.10) is the larger for $k \gg k_c$. Hence, neglecting the first term, and using the Kolmogorov expression for $E(k)$, valid in the range considered,

$$E(k) \approx \epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}, \tag{3.11}$$

we find, for the magnetic spectrum,

$$\Gamma(k) \approx \overline{H_j^2}\epsilon^{\frac{2}{3}}\lambda^{-2}k^{-\frac{11}{3}} \quad \text{for} \quad k_c \ll k \ll k_v. \tag{3.12}$$

This spectrum does fall off rapidly compared with the energy spectrum $E(k)$ in the same range as we were led to presuppose.

It is illuminating to compare this result with the work of Golitsyn (1960) who found a similar $k^{-\frac{11}{3}}$ law, modified by an anisotropic factor, for the spectrum of small-scale field variations induced by turbulent motion in a uniform applied field. He used a perturbation method, supposing that the field fluctuations are always small compared with the applied field, a situation that would seem to persist only when $R_m < 1$. In Golitsyn's case, the lines of force at any instant would be approximately straight with small fluctuations. In the case $R_m \gg 1$, the lines of force will be randomly oriented, and approximately uniform with small fluctuations through each region of fluid of dimension $l_c = (\lambda^3/\epsilon)^{\frac{1}{4}}$. The spectrum of the small-scale fluctuations within each such region could be determined by Golitsyn's method, introducing an anisotropic factor for each region. When we average over all the regions, the anisotropic factor disappears, and we are left with the isotropic law (3.12) that we have already determined by an independent method. The magnetic-field spectra in the two cases are sketched in figure 1.

4. Conclusions

Since the spectra (2.10) and (3.12) must agree in order of magnitude at $k = k_c$, and since the dimensionless constant must be chosen so that

$$\tfrac{1}{2}\overline{H^2} = \int_0^\infty \Gamma(k)\,dk, \tag{4.1}$$

they may be written in the form

$$\Gamma(k) = \tfrac{4}{9}\lambda\overline{H^2}\epsilon^{-\frac{1}{3}}k^{\frac{1}{3}} \qquad (L^{-1} \ll k \ll k_c), \tag{4.2}$$

$$\Gamma(k) = \tfrac{4}{9}\lambda^{-2}\overline{H^2}\epsilon^{\frac{2}{3}}k^{-\frac{11}{3}} \quad (k_c \ll k \ll k_v). \tag{4.3}$$

It may fairly be supposed that $\Gamma(k)$ does not behave too erratically for $k < L^{-1}$, and that it falls off very rapidly for $k > k_v$.

We may deduce a rough rule for computing the net effect of the turbulence on the magnetic field as follows. In the absence of the turbulence, only the large-scale magnetic field distribution $\mathbf{H}_0(\mathbf{r})$ would be present, with a mean-square value of approximately

$$\overline{H_0^2} = \int_0^{L^{-1}} \Gamma(k)\,dk \doteqdot L^{-1}\Gamma(L^{-1}). \tag{4.4}$$

If we suppose that the spectral law (4.2) is valid, at least in order of magnitude, right up to $k = L^{-1}$, then, using equations (1.2) and (2.5), equation (4.4) becomes

$$\overline{H^2} = \tfrac{9}{4}R_m\overline{H_0^2}, \tag{4.5}$$

indicating the extent to which turbulence at large magnetic Reynolds number increases the mean-square field intensity (provided always that $R_m < R$).

The extent to which the dissipation of energy by ohmic heating is increased may also be readily calculated. Thus if

$$D_0 = \lambda\int_0^{L^{-1}} k^2\Gamma(k)\,dk \doteqdot \lambda L^{-3}\Gamma(L^{-1}) \tag{4.6}$$

is the dissipation in the absence of turbulence and

$$D = \lambda \int_0^\infty k^2 \Gamma(k)\, dk \doteqdot \lambda k_c^3 \Gamma(k_c) \tag{4.7}$$

is the increased dissipation under turbulent conditions, then

$$D = R_m^{\frac{5}{2}} D_0. \tag{4.8}$$

This result may be interpreted in terms of an eddy conductivity σ_e, defined so that the total dissipation, by analogy with (4.6), is given by

$$D = (4\pi\mu\sigma_e L^3)^{-1}\, \Gamma(L^{-1}). \tag{4.9}$$

Then evidently σ_e may be expressed in the equivalent forms

$$\sigma_e = R_m^{-\frac{5}{2}} \sigma = (4\pi\mu u' L)^{-\frac{5}{2}} \sigma^{-\frac{3}{2}}, \tag{4.10}$$

indicating, first, the strong dependence of σ_e on R_m, and secondly, the anomalous effect, already noticed for a scalar spectrum, that increase of σ, keeping other parameters constant, induces a decrease in σ_e, simply because the conduction cut-off k_c is raised so that more thorough mixing of the magnetic field is possible.

As we stated at the outset, we have assumed throughout that the mean magnetic energy per unit mass $\mu \overline{H^2}/4\pi\rho$ must be small compared with the kinetic energy per unit mass of the small-scale motion. By dimensional reasoning, this latter quantity is of order of magnitude $(\epsilon\nu)^{\frac{1}{2}}$, which, by equations (1.1) and (2.5), is of order $R^{-\frac{1}{2}}$ times $\frac{1}{2}u'^2$, the total kinetic energy per unit mass, as stated in the introduction. Using (4.5), this places the following restriction on the magnitude of $\overline{H_0^2}$,

$$\mu \overline{H_0^2}/4\pi\rho \ll u'^2/R_m R^{\frac{1}{2}}. \tag{4.11}$$

It is not difficult, however, to visualize how our results must be modified if this condition is violated. The field, approximately uniform over a region of dimension k_c^{-1}, will tend to extinguish any eddies of smaller scale whose energy it exceeds (cf. the experiments of Murgatroyd 1953). Now the energy per unit mass of Fourier components of the motion of wave-number greater than k is approximately $\epsilon^{\frac{2}{3}} k^{-\frac{2}{3}}$. Let us define a wave-number k_f by the stipulation that the motion consisting of eddies of wave-number greater than k_f should have approximately the same energy density as the total magnetic field. If $k_f \gg k_v$, we have the situation already described wherein the applied field is very weak. At the other extreme, if $k_f \ll L^{-1}$, we have the case of a field strong enough to suppress the turbulence completely. In the intermediate case, when $L^{-1} \ll k_f \ll k_v$, so that k_f is given by

$$\mu \overline{H^2}/4\pi\rho = \epsilon^{\frac{2}{3}} k_f^{-\frac{2}{3}}, \tag{4.12}$$

the effect of the field will be to suppress eddies of wave-numbers large compared with k_f. If $k_f > k_c$, both the kinetic and therefore magnetic energy spectra may thus be expected to fall off very rapidly for $k > k_f$. If $k_f < k_c$, equipartition of energy between kinetic and magnetic modes will probably be established at all wave-numbers greater than k_f, both spectra falling off rapidly in the range of ohmic dissipation, $k > k_c$. These observations are sketched schematically in figure 2.

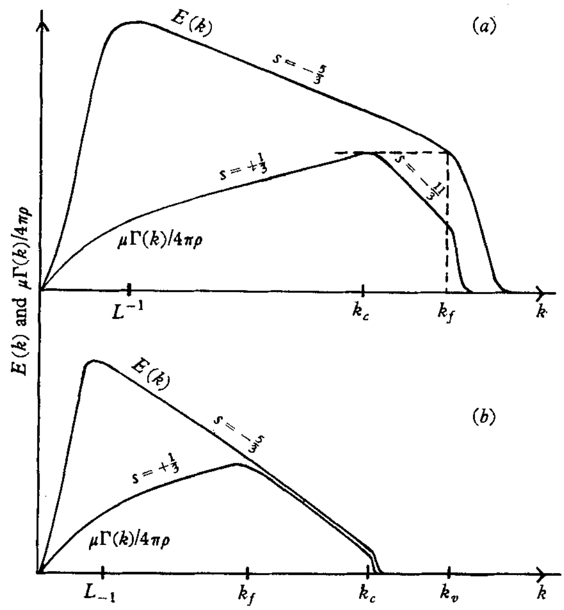

FIGURE 2. Schematic illustration of the possible effect of magnetic stabilizing forces on kinetic and magnetic spectra in the two cases: (*a*) $k_f > k_c$, (*b*) $k_f < k_c$.

It is a pleasure to record my thanks to Dr G. K. Batchelor for the stimulating discussions that I have enjoyed with him on the subject of this paper.

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